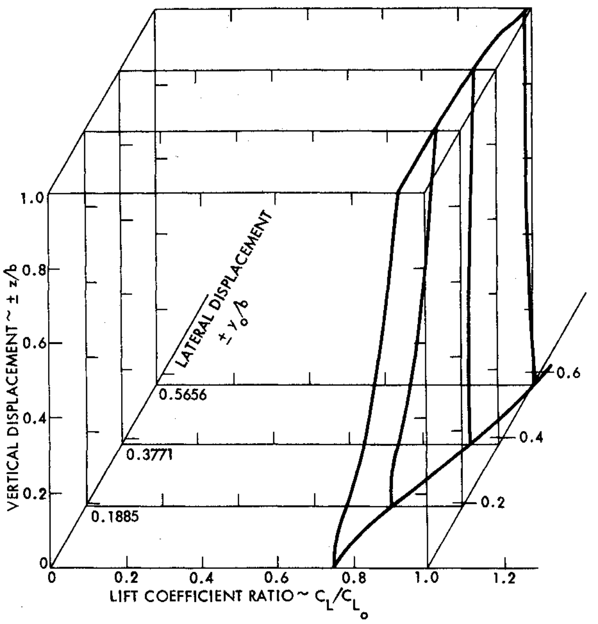

Fig. 3 Trailing aircraft lift coefficient variation with vertical and lateral displacement.

Equations (1) and (2) give the change in lift and rolling moment coefficients of the trailing aircraft wing from the undisturbed equilibrium flight condition (i.e., away from the influence of the leading aircraft trailing vortices). $C(L)_O$ is the lift coefficient of the leading aircraft. The coordinates z and y correspond to the distance of the centerline of the trailing wing below and to the right of the midpoint between the leading aircraft's two trailing vortices. The coordinate x is the longitudinal distance between the leading and trailing aircraft. The primes refer to the local position on the trailing wing with respect to its centerline. AR_t and AR_l are the trailing and leading wing aspect ratios, m the trailing wing lift-curve slope, S, $\bar{c}$, and b are the leading wing's area, mean aerodynamic chord, and wing-span, respectively.

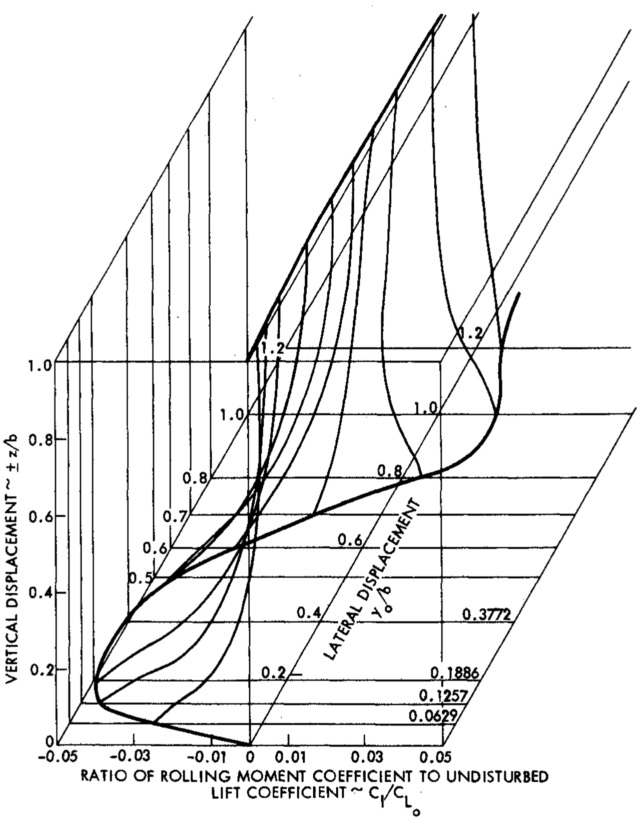

Fig. 4 Trailing aircraft rolling moment variation.

Values of lift and rolling moment coefficient, calculated from Eqs. (1) and (2), are shown in Figs. 3 and 4. These values were used in a three-degree-of-freedom analog computer program simulating the trailing aircraft motion. Results of these simulations which indicate the very erratic motion that can occur have been reported in Ref. 3.

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Theoretical Suction and Pressure Distribution Bounds for Flow Separation in Retarded Flow

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Nomenclature

A = damping-length constant, see Eq. (6b)
b = parameter, see Eq. (8)
c_f = local skin-friction coefficient, see Eq. (15)
f = dimensionless stream function
L = modified mixing length, see Eq. (6a)
P = pressure-gradient parameter, $(x/u_e)/(du_e/dx)$
$P+$ = dimensionless pressure-gradient parameter, $-(dp/dx)\,\nu/\rho u^3$
R_x = Reynolds number, $u_e x/\nu$
u, v = x- and y-components of velocity, respectively
$v_w{}^+$ = dimensionless mass transfer parameter, v_w/u_τ
u = friction velocity, $(\tau_w/\rho)^{1/2} = u_e(c_f/2)^{1/2}$
x, y = rectangular coordinates
ϵ = eddy viscosity
ϵ^+ = dimensionless eddy viscosity, ϵ/ν
η = transformed y-coordinate, see Eq. (9)
μ = dynamic viscosity
ν = kinematic viscosity
ρ = density
τ = shear stress
ψ = stream function

Subscripts

e = outer edge of boundary layer
i = inner region
0 = outer region
w = wall

primes denote differentiation with respect to η.

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Introduction

CONTROLLING the boundary layer developed by a retarded flow over a surface is an important design problem in many areas of fluid mechanics. The combination of an adverse pressure gradient impressed on the boundary layer and the viscous forces acting within the boundary layer may result in a local reversal of the flow at the wall. When this occurs, the boundary layer separates from the surface and changes the external pressure distribution drastically with resultant losses in performance for such devices as airfoils and diffusers. In the case of airfoils, where it is desired to maximize the lift, it is necessary to compute the minimum distance over which a given pressure rise can be obtained without the flow separating. The most rapid pressure rise that it is possible to obtain occurs when the wall shear stress (τ_w) along the suction side of the airfoil decreases to zero. Therefore it is of considerable interest to be able to calculate boundary layers with specified values for the wall shear that also decrease to zero. Mathematically this leads to a form of inverse problem where the pressure gradient in the governing equations must be determined such that the solution satisfies an overdetermined set of boundary conditions. Boundary layers where both the pressure distribution and the wall shear are specified may also be calculated by allowing suction at the wall. Unlike the previous condition however, this is not an inverse type problem but involves a direct solution of the standard boundary-layer equations.

In this Note we show an efficient method for calculating the boundary layers subject to both of these conditions. To illustrate the method, we determine the theoretical suction and pressure distribution bounds for boundary layer separation in retarded turbulent flows for two Reynolds numbers, $R_L = 10^6$ and $R_L = 10^7$. We first consider the calculation of the external pressure distribution required to produce constant skin-friction coefficients of 2×10^{-3}, 1×10^{-3}, and 5×10^{-4} along the surface. The case of a zero skin-friction coefficient would correspond to the limiting condition where the flow is on the verge of separating. The cases evaluated here for $c_f \neq 0$ represent flows with varying margins of safety with respect to separation, depending on the magnitude of the specified skin-friction that is considered.

We also consider the suction bounds for boundary layer separation in retarded flows. We specify the external velocity distribution and calculate the suction requirements for the same skin-friction coefficients considered above. The suction velocity for a zero skin-friction coefficient would correspond to the optimum (minimum) suction velocity necessary to keep the flow attached. The cases considered here again represent flows with varying margins of safety with respect to separation.

Basic Equations

The boundary-layer equations for a two-dimensional incompressible turbulent flow are:
Continuity

$$(\partial u/\partial x) + (\partial v/\partial y) = 0 \tag{1}$$

Momentum

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u_e\frac{du_e}{dx} + \frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y} - \rho\overline{u'v'}\right) \tag{2}$$

The boundary conditions are:

$$y = 0 \qquad u = 0, \qquad v = v_w \tag{3a}$$

$$y = \delta \qquad u = u_e \tag{3b}$$

To satisfy the closure conditions in Eq. (2) we use the eddy viscosity concept, and represent the Reynolds shear stress term by

$$-\rho\overline{u'v'} = \rho\epsilon\frac{\partial u}{\partial y} \tag{4}$$

We adopt the eddy viscosity formulation of Refs. 1 and 2 and define ϵ by two separate expressions. For two-dimensional incompressible flows ϵ is written as

$$\epsilon = \begin{cases} \epsilon_i = L^2\left|\frac{\partial u}{\partial y}\right|, \quad \epsilon_i \le \epsilon_0 & \text{(5a)} \\ \epsilon_0 = 0.0168\left|\int_0^\infty (u_e - u)dy\right|, \quad \epsilon_0 \ge \epsilon_i & \text{(5b)} \end{cases}$$

where

$$L = 0.4y[1 - \exp(-y/A)] \tag{6a}$$

$$A = 26\nu(u_\tau)^{-1/2}/N \tag{6b}$$

$$N = \left[\frac{p^+}{v_w^+}\{1 - \exp(11.8v_w^+)\} + \exp(11.8v_w^+)\right]^{1/2} \tag{6c}$$

Introducing the relation (4) into (2), we can write the momentum equation as

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u_e\frac{du_e}{dx} + \nu\frac{\partial}{\partial y}\left[(1+\epsilon^+)\frac{\partial u}{\partial y}\right] \tag{7}$$

Equations (1) and (7), expressed in physical coordinates, are singular at $x = 0$. For this reason, we first transform them to a coordinate system that removes the singularity and stretches the coordinate normal to the flow. We define the stream function by

$$u = (\partial\psi/\partial y), \qquad v = -(\partial\psi/\partial x) \tag{8}$$

and introduce the similarity parameter η by

$$\eta = \left(\frac{u_e}{\nu x}\right)^{1/2} y \tag{9}$$

together with a dimensionless stream function $f(x,\eta)$ defined by

$$\psi = (u_e\nu x)^{1/2} f(x,\eta) \tag{10}$$

With the relations given by Eqs. (8–10), we can write the momentum equation as

$$(bf'')' + \frac{P+1}{2}ff'' + P[1 - (f')^2] = x\left(f'\frac{\partial f'}{\partial x} - f''\frac{\partial f}{\partial x}\right) \tag{11}$$

Similarly the boundary conditions in Eq. (3) can be written as

$$\eta = 0 \qquad f = f_w \qquad f' = 0 \tag{12a}$$

$$\eta = \eta_\infty \qquad f' = 1 \tag{12b}$$

In terms of transformed variables, the inner and outer eddy viscosity formulas in Eq. (5) can be written in dimensionless form as

$$\epsilon^+ = \begin{cases} \epsilon_i^+ = 0.16R_x^{1/2}\eta^2\,|f''|\,[1 - \exp(-y/A)]^2 & \epsilon_i^+ \le \epsilon_0^+ \\ \epsilon_0^+ = 0.0168R_x^{1/2}[\eta_\infty - f(\eta_\infty)] & \epsilon_0^+ \ge \epsilon_i^+ \end{cases} \tag{13}$$

where y is given by Eq. (9) and A by Eq. (6b).

The Inverse Problem and Newton's Method

The system given by Eqs. (11–14) with specified $u_e(x)$ or $P(x)$ is the typical two-dimensional boundary-layer problem; for brevity we shall call it the *standard problem.*

The *inverse problem* results from requiring that the wall shear be specified, that is,

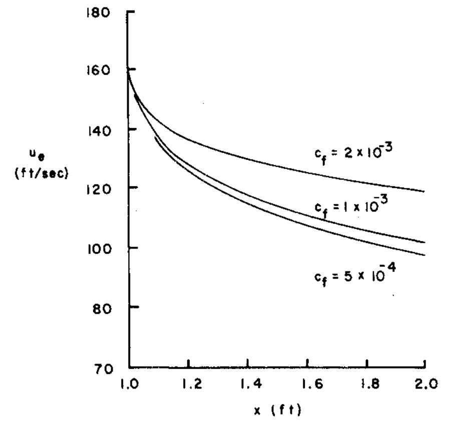

Fig. 1 Computed external velocity distribution for $R_L = 10^6$. $x = 1$ corresponds to the beginning of the adverse pressure gradient.

$$\tau_w(x) = \mu \left(\frac{\partial u}{\partial y} \right)_w \tag{14}$$

It is more convenient to express the wall shear in terms of dimensionless quantities. For this reason we normalize τ_w by the local dynamic pressure $\frac{1}{2} \rho u_e^2$ and write it as

$$(\tau_w/\rho)u_e^{-2}(x) = (c_f/2) \tag{15}$$

In terms of transformed coordinates Eq. (15) becomes

$$(c_f/2) = [f_w''/(R_x)^{1/2}] \tag{16}$$

Here c_f is the local skin-friction coefficient. Thus specifying the local dimensionless wall shear is the same as specifying the local skin-friction coefficient. The system given by Eqs. (11-13) and by Eq. (16) is overdetermined and we cannot specify $P(x)$ or $u_e(x)$ arbitrarily. Rather, we must determine $P(x)$ as well as $f(x,\eta)$ to solve the system. To describe our approach to the problem, let us assume that at $x = x_{n-1}$, we are given the initial profiles, namely, f, f', f'' and the pressure gradient $P(x_{n-1})$ and velocity $u_e(x_{n-1})$, and that at $x = x_n$ we seek the solution of Eq. (11) subject to Eq. (13) and subject to a given $c_f(x)$. To start the calculations, it is necessary to know $P(x)$ and $u_e(x)$. The latter is necessary since R_x is a function of u_e. Here we first assume $P(x)$ and calculate $u_e(x)$ from the definition of $P(x)$, namely,

$$P = (x/u_e)(du_e/dx)$$

In terms of central differences we can write the above expression for $(u_e)_n$ as

$$(u_e)_n = -(u_e)_{n-1} \left[\frac{P_{n-1/2} + 2\alpha_n}{P_{n-1/2} - 2\alpha_n} \right] \tag{17}$$

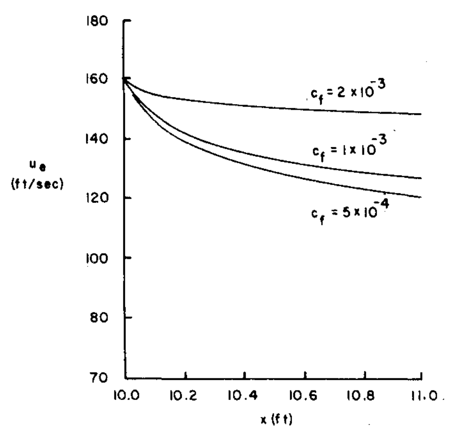

Fig. 2 Computed external velocity distribution from $R_L = 10^7$. $x = 10$ corresponds to the beginning of the adverse pressure gradient.

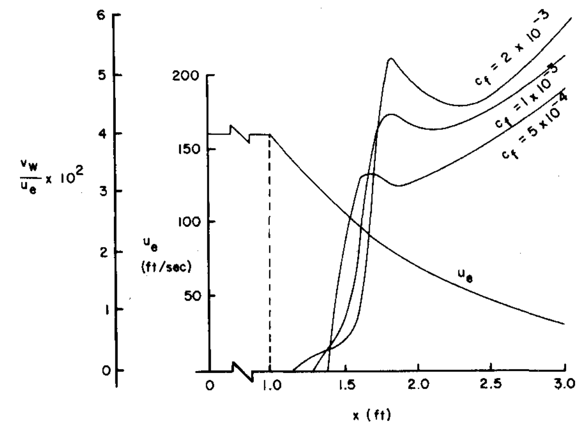

Fig. 3 Computed suction velocity distribution for $R_L = 10^6$.

where

$$\alpha_n = \frac{x_{n-1/2}}{x_n - x_{n-1}}, \quad P_{n-1/2} = \frac{1}{2}(P_n + P_{n-1}),$$
$$x_{n-1/2} = \frac{1}{2}(x_n + x_{n-1}) \tag{18}$$

Once $P(x)$ and $u_e(x)$ are known, then the standard problem, namely, Eq. (11) subject to Eq. (12), can be solved. The numerical method used is described in ref. 3. Let us denote the solution of the standard problem by

$$f(x,\eta) = g(x,\eta,P(x)) \tag{19}$$

Using that solution, we can now calculate c_f from Eq. (16). Let us denote it by c_f and denote the specified c_f by $S(x)$. Then we can form

$$\phi(P(x)) \equiv c_f - S(x) \qquad x > 0 \tag{20}$$

and seek $P(x)$ such that $\phi = 0$ on $x > 0$.

To solve $\phi = 0$ we use Newton's method. Thus with some estimate $P^{(0)}(x)$ of the desired pressure gradient, we define the sequence $\{P^{(\nu)}(x)\}$ by setting

$$p^{\nu+1}(x) = p^{\nu}(x) - \frac{\phi(P^{(\nu)}(x))}{\frac{\partial}{\partial P}[\phi(P^{(\nu)}(x))]} \tag{21}$$

The derivative of ϕ with respect to P can be obtained from Eq. (20) by making use of the relation given by Eq. (16). This gives

$$\frac{\partial \phi}{\partial P} = \frac{2}{(R_x)^{1/2}} \left[-F_w'' + f_w'' \frac{(u_e)_{n-1}}{(u_e)_n} \left(\frac{2\alpha_n}{(P_{n-1/2} - 2\alpha_n)^2} \right) \right] \tag{22}$$

where

$$F_w'' = (\partial f_w''/\partial P) \tag{23}$$

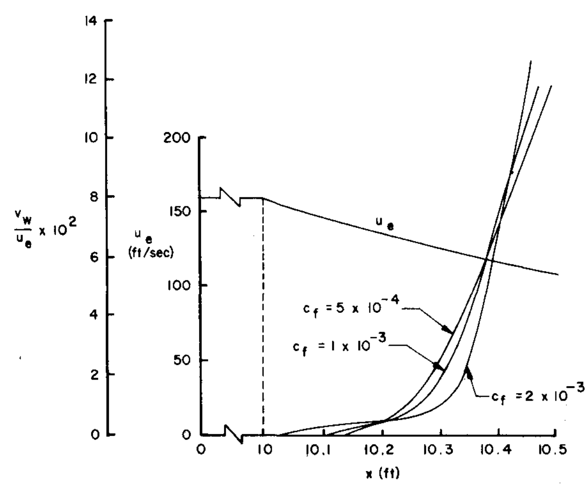

Fig. 4 Computed suction velocity distribution for $R_L = 10^7$.

To summarize one step of iteration of Newton's method, we first assume $P(x)$, calculate u_e from Eq. (17), and obtain a solution of Eq. (11) subject to (12). The solution yields a $c_f^{(\nu)}$ according to Eq. (16). From this result and from the specified c_f [$=S(x)$], we find ϕ from Eq. (20). It is then clear that the next value of $P(x)[\equiv P^{\nu+1}(x)]$ can be calculated from Eq. (21), provided that $\partial\phi/\partial P$ is known. Its calculation is discussed in Ref. 4. Once it is calculated, then a new value of $P(x)$ is obtained from Eq. (21), and the standard problem is solved again. The iteration process is repeated till

$$\left|P^{(\nu+1)}(x) - P^{(\nu)}(x)\right| < \gamma_1 \tag{24}$$

where γ_1 is a convergence parameter.

Results

The method discussed in the previous sections was used to compute several examples to determine the theoretical suction and pressure distribution bounds for boundary layer separation in retarded turbulent flows for $R_L = 10^6$ and $R_L = 10^7$. Here R_L is defined as

$$R_L = (u_e L/\nu)$$

The distance L denotes the beginning of the adverse pressure gradient. The solution of the standard problem and the inverse is obtained by using the method of Ref. 3.

Figures 1 and 2 show the external pressure distribution required to produce constant skin-friction coefficients of 2×10^{-3}, 1×10^{-3}, and 0.5×10^{-3} along the surface. In these calculations the boundary layers were developed to the desired Reynolds number, R_L, by turbulent flow with zero-pressure gradient. The flow at L was then rapidly decelerated at a linear rate until the desired skin friction coefficient was obtained. The inverse problem was then solved to determine the required external velocity distribution to maintain the desired c_f-distribution.

Figures 3 and 4 show the suction velocity distributions required to produce constant skin-friction coefficients of 2×10^{-3}, 1×10^{-3}, and 0.5×10^{-3} along the surface. In these calculations the boundary layers were again developed to the desired Reynolds numbers by flow over a flat plate. The flow was then decelerated at an exponential rate and the desired suction velocities were computed by using the standard problem. For details see Ref. 4.

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